THE SPECTRUM OF NORMAL OSCILLATIONS OF AN ELASTOPLASTIC MEDIUM WITH DISSIPATION

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1. Introduction. A considerable number of publications have recently apepared in which the ideas of gauge field theories have been used to describe the dynamics of defects in condensed media (see, for example, [1-5]). Unlike the theory of elasticity, in theories which describe the mechanics of the deformation of a solid with defects, in addition to the so-called external degrees of freedom (the role of which may be played by the total displacements of points of the body $u_r(r)$), a certain number of internal degrees of freedom (depending on the model) are also introduced. In the simplest case, by considering only translational defects, one can choose as the internal degrees of freedom (at each point) nine components of plastic distortion β_{ij}. Using chosen variables, a gauge-invariant Lagrangian [6] is described, the variation of which leads to dynamic equations of the model of the medium considered. It is important to note that, since the equations of motion of the medium are obtained from the principle of least action with a Lagrangian which does not depend explicitly on time and is invariant under global rotations and displacements of the system of coordinates, the law of conservation of mechanical energy, momentum and angular momentum are automatically satisfied, and the system is therefore strictly nondissipative. The absence of dissipation is one of the main reasons why it is difficult to give a physical interpretation of the theory as a model of an elastoplastic medium and to compare it with experiment.

As was shown in [1], in the framework of this theory one can nevertheless speak of dissipation if one means by it the process by which energy is transferred from some degrees of freedom, which we will consider to be external ones, to other degrees of freedom of this medium, which we will regard as internal. This energy exchange between degrees of freedom must occur in any model. In an extended system with energy exchange at the boundary, this process will be irreversible, and will thereby have all the features of energy dissipation. It is clear, however, that this approach to dissipation does not exhaust all possible channels of energy irreversibility. The problem is whether one can describe the main channels of dissipation which occur in actual media, or whether they remain outside the framework of this model. In our opinion, the main dissipation channel is neglected in this model and requires a special investigation. It is obvious that the dissipation that occurs in elastic motions of the medium is considerably less than that which occurs in plastic deformation. The latter is related to acoustic radiation when the defects surmount the Peierls barriers and other processes which only occur in a discrete medium, which any real medium is. Such processes are generally not considered in gauge theories of mechanics, i.e., the main dissipation channel is ignored.

In this paper we consider dissipation connected with plastic degrees of freedom. Dissipation is introduced explicitly by appropriately including certain dissipative terms in the equations of motion. The dissipation is assumed to be fairly small. It can therefore be described by a so-called dissipative function [7]. The system of equations of motion of this model are linear and enable one to determine completely the dispersion relations for normal oscillations, which, when dissipation is introduced, turn out to be complex. A similar procedure has already been used in [8, 9] for media without dissipation.

2. The Lagrangian and the Dissipative Function. We will start from the gauge Lagrangian in the simplest "minimum coupling" model [6], which has already been used in [8, 9]:

$$L_{el-pl}=\int dV\left\{\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}-\mu\left(\frac{\partial u_i}{\partial x_k}-\beta_{ki}\right)\left(\frac{\partial u_i}{\partial x_k}-\beta_{ki}\right)-\mu\left(\frac{\partial u_i}{\partial x_k}-\beta_{ki}\right)\times\right.$$
$$\left.\times\left(\frac{\partial u_k}{\partial x_i}-\beta_{ik}\right)-\lambda\left(\frac{\partial u_i}{\partial x_i}-\beta_{ii}\right)\left(\frac{\partial u_k}{\partial x_k}-\beta_{kk}\right)+B\frac{\partial\beta_{km}}{\partial t}\frac{\partial\beta_{km}}{\partial t}-C\alpha_{km}\alpha_{km}\right\}. \tag{2.1}$$

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Here $u_i(x)$ is the displacement vector of a point with initial coordinate x, ρ is the density of the medium, λ, μ are the Lamé coefficients, dV is the volume differential, β_{ij} is the plastic-distortion tensor, B and C are new material components, and $\alpha_{km} = e_{kij}\partial\beta_{jm}/\partial x_i$ is the dislocation-density tensor. In (2.1) and henceforth summation is understood to be carried out over repeated indices. It is also assumed that the medium is incompressible for plastic deformation

$$\mathrm{Sp}\,\beta_{ij} = \beta_{kk} = 0, \tag{2.2}$$

which is a good approximation for actual solids.

Energy dissipation can be taken into account by adding a friction force to the equations of motion obtained using Lagrangian (2.1). As we know [7], for a small value of the dissipation the friction forces are linear functions of the generalized velocities (in this case $\partial\beta_{ij}/\partial t$), and can be determined using the dissipative function R. The latter must be a scalar quadratic function of the tensor $\partial\beta_{ij}/\partial t$. For a qualitative analysis we will confine ourselves to the simplest form of this function, namely,

$$R = \eta\frac{\partial\beta_{ij}}{\partial t}\frac{\partial\beta_{ij}}{\partial t}. \tag{2.3}$$

3. <u>The Equations of Motion and the Dispersion Relations</u>. The equations of motion for any generalized coordinate are the Euler–Lagrange equations, on the right-hand side of which viscous forces are added

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = -\frac{\partial R}{\partial \dot{q}}.$$

Bearing (2.1)-(2.3) in mind, we obtain the dynamic equations

$$\rho\frac{\partial^2 u_p}{\partial t^2} - \lambda\frac{\partial^2 u_q}{\partial x_p \partial x_q} - \mu\left(\frac{\partial^2 u_p}{\partial x_q \partial x_q} + \frac{\partial^2 u_q}{\partial x_q \partial x_p}\right) + \mu\left(\frac{\partial\beta_{pq}}{\partial x_q} + \frac{\partial\beta_{qp}}{\partial x_q}\right) = 0, \tag{3.1}$$

$$B\frac{\partial^2\beta_{pa}}{\partial t^2} - C\left(\frac{\partial^2\beta_{pq}}{\partial x_k \partial x_k} - \frac{\partial^2\beta_{iq}}{\partial x_j \partial x_p}\right) - \lambda\frac{\partial u_i}{\partial x_i}\delta^{pq} - \mu\left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\right) + \mu(\beta_{pq} + \beta_{qp}) + 2\eta\frac{\partial\beta_{pq}}{\partial t} - \gamma\delta^{pq} = 0$$

(γ is an undetermined Langange multiplier). We will seek solutions in the form

$$u_p,\ \beta_{pq} \sim \exp(-i\omega t + ikr).$$

Substituting these into the system of equations (3.1) and solving the characteristic equation obtained we obtain eleven characteristic frequencies for each value of k. Two forms of representing the dispersion relations are possible. If we choose k to be real, the frequencies turn out to be complex, and the imaginary part of the frequency gives a quantity which is the inverse of the attenuation time of the corresponding normal mode. If the real quantities ω are specified, the wave vectors turn out to be complex, and the imaginary part of the wave vector will determine the inverse depth of penetration of the excitation of this frequency into the medium. Bearing in mind subsequent applications to the propagation of waves in the medium, we will choose the second representation. The solutions of the characteristic equation of system (3.1) for k can be obtained analytically. Even in the case of zero dissipation they are given by lengthy expressions, which we will not reproduce here. The results of numerical calculations, which demonstrate the changes in the spectrum of normal oscillations, taking dissipation into account, are shown in Figs. 1-3.

4. <u>The Configuration of the Normal Modes in the Case of Zero Dissipation</u>. In Fig. 1 we show dispersion curves for the case of zero dissipation (η = 0). The constants of the material are chosen to be the same as in [8] (λ = 2, μ = 1, B = 2/9, C = 1/72, ρ = 1), in which the possibility of determining them experimentally is discussed. It can be seen that of the eleven branches of normal oscillations, two are zero, four correspond to acoustic oscillations (ω → 0 when k → 0), and five are optical oscillations ($\omega \to \omega_0 \neq 0$ when k → 0). The nine nonzero modes define different forms of excitations which propagate in the crystal, while the two zero modes define the remaining deformations.

An analysis of the configurations of the normal oscillations shows that in the long-wave limit (k → 0) the optical modes $k_{3,4}$, $k_{5,7}$ and k_{10} are oscillations of the internal degrees

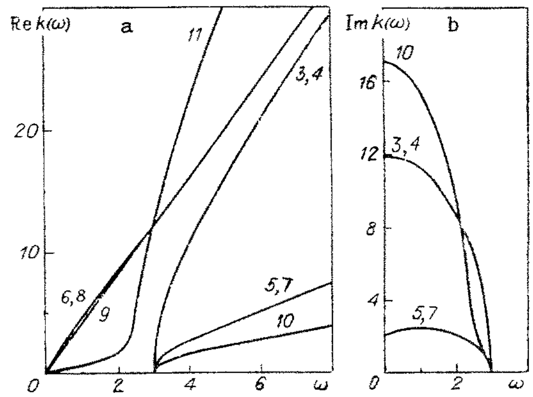

Fig. 1

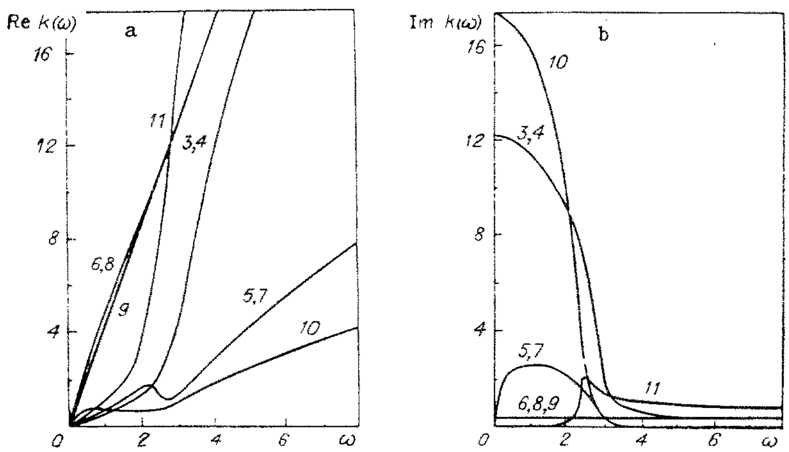

Fig. 2

of freedom with a small addition of external modes ($u \sim k\beta$). The mode k_9 corresponds to oscillations of the internal degrees of freedom ($u_i = 0$, $\beta_{ij} \neq 0$) for any k. The oscillations of the external degrees of freedom as $k \to 0$ are modes $k_{6,8}$ and k_{11}. However, for these branches the addition of plastic modes is not small (the elastic and plastic distortions are of the same order of magnitude). Note that the law of dispersion of the mode k_{11} as $k \to 0$ has the form

$$k_{11}^2 = \frac{\rho}{K}\,\omega^2,$$

where $K = \lambda + 2/3\,\mu$ is the bulk modulus of the elastic continuum. There are no transverse acoustic oscillations as $\omega \to 0$. Hence, the spectrum of the medium considered in the long-wave limit exhibits the features of the spectrum of an ordinary liquid. In the long-wave limit the spectrum of the medium in question shows a similarity with the spectrum of a solid. Thus, as $\omega \to \infty$ the modes $k_{5,7}$ and k_{10} describe purely elastic oscillations ($\beta \sim u/k$), which also manifests itself in the dispersion laws: as $\omega \to \infty$ they are linear and correspond to the propagation of waves with the velocities of longitudinal sound $C_{\parallel} = \sqrt{(2\mu + \lambda)/\rho}$ and transverse sound $C_{\perp} = \sqrt{\mu/\rho}$ in an elastic medium. The remaining modes, in the limit of large k, describe oscillations of the internal degrees of freedom. The analogy with the spectrum of a liquid as $\omega \to 0$, and of a solid as $\omega \to \infty$ becomes particularly close when energy dissipation is taken into account.

5. Discussion of the Results. The value of the dissipation in the medium will depend on the ratio of the elastic and viscous forces in the equations of motion (3.1). Comparing the dynamic terms $\mu\beta_{pq}$ with the viscous term $\eta\partial\beta_{pq}/\partial t$ and assuming that $\partial\beta_{pq}/\partial t \sim \omega_0\beta_{pq}$, we find the dissipative force is governed by the dimensionless parameter

$$\varkappa = \frac{\eta\omega_0}{\mu} = \frac{\eta}{(\mu B)^{1/2}}.$$

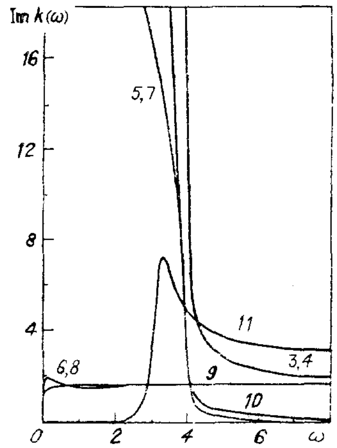

Fig. 3

Figure 2 shows the form of the dispersion curves when $\kappa = 0.05$. The remaining constants are the same as in Fig. 1. It is noteworthy that the inclusion of dissipation has the least effect on the short-wave part of the spectrum. In the low-frequency region a qualitative readjustment of the spectrum occurs even for very low dissipation. Since the low-frequency region is subjected to the most intense rearrangement, in Fig. 3 we show curves of $\mathrm{Im}k(\omega)$ separately for $\kappa = 0.02$ in the region $\omega \ll \omega_0$. When dissipation is included the former acoustic branches 6, 8, 9 and 11 become complex for any ω. The attenuation of $k_{6,8}$ and k_9 is independent of the frequency and is determined by the coefficient of viscosity. The branch k_{11}, which up to a certain frequency (of the order of $\omega_0 = (2\mu/B)^{1/2}$) is a weakly attenuating branch ($\mathrm{Im}k_{11} \sim \omega^2$), is of particular interest. Then a sharp bend is observed on the $\mathrm{Im}k_{11}(\omega)$ curve, after which the curve approaches a certain constant limit, which is nonzero, and hence the perturbations of this branch will attenuate at a certain depth.

We will now consider the optical branches $k_{3,4}$, $k_{5,7}$ and k_{10}. When there is no dissipation the wave vectors of these branches are purely real when $\omega > \omega_0$ and purely imaginary when $\omega < \omega_0$ (Fig. 1a, b). When dissipation is included, in the region of low values of ω the optical branches, as before, are decaying branches, while in the high-frequency limit three branches ($k_{5,7}$ and k_{10}), corresponding to longitudinal sound and two polarizations of transverse sound, describe the propagation of perturbations without attenuation. Hence, the dispersion relations shown in Figs. 1 and 2 enable one to follow in detail the change in the response of the medium to a periodic external force from liquid-like in the low-frequency limit to solid-like in the high-frequency limit.

The inclusion of dissipative terms in the equation of motion in the model of the medium considered leads to a considerable change in the spectrum of normal oscillations. The frequency of the optical oscillations ω_0 remains the characteristic frequency of the medium. When the frequency of the inducing forces in the medium in question changes for $\omega \ll \omega_0$, only one weakly decaying branch will be excited, corresponding to compression waves in a liquid. In the opposite limit ($\omega \gg \omega_0$), three branches, the configuration of which corresponds to longitudinal and transverse sound in an elastic medium, are weakly decaying. Hence, at frequencies below a certain characteristic frequency ω_0 the medium behaves as a liquid, and at higher frequencies it behaves as a solid.

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NONLINEAR BENDING OF TOROIDAL SHELLS OF ARBITRARY TRANSVERSE CROSS SECTION LOADED WITH INTERNAL PRESSURE

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In this paper we derive complete geometrically nonlinear relations for the problem of the bending of a toroidal shell of arbitrary transverse cross section. An accurate expression is obtained for the potential of the internal-pressure forces, which holds for any distortions of the shape of the cross section. An algorithm and a numerical solution of the problems of the deformations of cylindrical and toroidal shells for large elastic displacements are considered. The results obtained are compared with existing analytic solutions and experimental data.

1. Introduction. Since the publication of the papers by Dubyaga [1] and Karman [2] there have been numerous investigations of the problem of the bending of thin-walled curvilinear tubes, most of which have been carried out using the linear theory of shells. The case of the combined action of internal pressure and bending moments on a tube of circular transverse cross section is considered in [3, 4] using variational principles. Another approach was employed in [5], which consists of solving the differential equations of the bending of a toroidal shell, first loaded with internal pressure. It was established that the stiffness properties and the stresses in the shell depend nonlinearly on the pressure. Small displacements were investigated and the problem was regarded as being linear in the bending moments. Large displacements for pure bending of cylindrical shells were considered in [6], and the value of the limiting moment for which a loss of the stability for the shells occurs was found, and the stability of a shell on bending, taking into account changes in its shape in the subcritical state was investigated for the first time. The results obtained in [6] were refined in [7-9] both by retaining small terms in the initial relations, and by choosing different approximating functions. The effect of the internal pressure when cylindrical shells are bent was taken into account in [10]. The previous results were generalized in [11] and two problems previously considered separately, were combined: the bending of curvilinear tubes in the linear formulation, and the deformation of cylindrical tubes in the case of large elastic displacements. The nonlinear equations of the bending of tubes with a small initial curvature of the axial line were derived and integrated approximately. The problem of the bending of curvilinear tubes loaded with an internal pressure was also solved in [12, 13], taking the geometrical nonlinearity into account.

It should be noted that the solutions mentioned above, particularly the nonlinear ones, were obtained using simplified deformation relations and retaining a small number of terms of the approximating series. It is therefore of interest to obtain more accurate results, particularly in the supercritical region.

Certain problems of the finite bending of curvilinear tubes were investigated in [14] using the nonlinear theory of shells.

2. Formulation of the Problem. We will regard the tube as a thin-walled toroidal shell. Suppose the tube is loaded with an internal pressure and boundary bending moments

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